UTILITY FUNCTIONS AND THE 'lin' OPERATION FOR CONVEX SETS

BY

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ABSTRACT

For an $\aleph_0$-dimensional space E of alternatives, there is described a preference relation $\succsim$ such that (in a very strong sense) *no* information about $\succsim$ can be expressed in terms of finite-dimensional linear transformations of E. The same construction shows that for each countable ordinal β, E contains a convex cone K such that $\text{lin}^{\beta} K = E$ but $\text{lin}^{\alpha} K \neq E$ for $\alpha < \beta$.

This note contributes to utility theory by shrapening a recent example of Kannai [5] and Perles, and to the geometry of infinite-dimensional convex sets by settling a problem raised in [11] concerning the iteration of the 'lin' operation. The reduction of the first matter to the second is described in Section 1 below, and the actual construction appears in Section 2. Section 3 mentions two unsolved problems.

1. **Utility theory.** An $\aleph$-*dimensional preference relation* is a transitive and reflexive relation $\succsim$ on an $\aleph$-dimensional real linear space E such that the following conditions are satisfied for all x, y and $z \in E$:

(1) if $x \succsim y$, then $x + z \succsim y + z$;
(2) if $x \succsim y$ and $\lambda > 0$, then $\lambda x \succsim \lambda y$;
(3) if $x \succsim kz$ for all positive integers k, then not $z \succ 0$.

The preference relation is called *pure* provided it satisfies the follow ng condition:

(4) if $x \sim 0$, then $x = 0$.

Here $x \succ y$ (x is *preferred* to y) means that $x \succsim y$ but not $y \succsim x$, while $x \sim y$ (x is *indifferent* to y) means that $x \succsim y$ and $y \succsim x$.

A *convex cone* is a set K such that $K + K \subset K$ and $]0, \infty[K \subset K$. Now consider a relation $\succsim$ on a (real) linear space, and let $S = \{x : x \succsim 0\}$. If the relation $\succsim$ is transitive and reflexive and satisfies conditions (1) and (2), then S is a convex cone with $0 \in S$, and

(5) $x \succsim y$ if and only if $x - y \in S$.

Conversely, if S is a convex cone with $0 \in S$ and the relation $\succsim$ is defined by (5), then $\succsim$ is transitive and reflexive and satisfies conditions (1) and (2).

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For a subset X of a linear space, lin X or $\mathrm{lin}^1 X$ will denote the union of X with the set of all endpoints of line segments contained in X. We then define $\mathrm{lin}^2 X = \mathrm{lin}(\mathrm{lin}^1 X), \cdots, \mathrm{lin}^\beta X = \mathrm{lin}(\mathrm{lin}^{\beta-1} X)$ if $\beta - 1$ exists, and $\mathrm{lin}^\beta X = \cup_{\alpha<\beta} \mathrm{lin}^\alpha X$ if β is a limit ordinal.

PROPOSITON. *Suppose that S is a convex cone in a linear space E, with $0 \in S$. Let the relation $\gtrsim$ be defined by* (7), *and let $T = \{x : x \succ 0\}$. Then the following four assertions are equivalent*:

(i) *the relation $\gtrsim$ satisfies condition* (3);
(ii) $(-S) \cap \mathrm{lin}\, S \subset S$;
(iii) $(-T) \cap \mathrm{lin}\, S = \emptyset$;
(iv) $(-T) \cap \mathrm{lin}\, T = \emptyset$.

Proof. Suppose first that condition (3) holds, and consider an arbitrary point $x \in -S \cap \mathrm{lin}\, S$. From the definition of lin S it follows that $]x, s] \subset S$ for some $s \in S$, whence for each positive integer k we have $(1-(1/k))x + (1/k)s \in S$. But this implies that $(s - x) + kx \in S$, whence $s - x \gtrsim k(-x)$ and consequently (by (3)) not $-x \succ 0$. But $-x \gtrsim 0$ (for $-x \in S$), and thus not $-x \succ 0$ implies $-x \sim 0$, whence $x \sim 0$ and $x \in S$. Thus (i) implies (ii).

If (ii) holds, then for each $x \in (-T) \cap \mathrm{lin}\, S$ we have $-x \succ 0$ (since $x \in -T$) and $x \gtrsim 0$ (since $x \in S$ by (ii)), whence $-x \succ 0 \gtrsim -x$. But this is impossible, so no such x exists and (ii) implies (iii). Obviously (iii) implies (iv).

Finally, let us suppose that (iv) holds, and consider points x and z of E such that $x \gtrsim kz$ for all positive integers k. For all k, we have $x \gtrsim (k+1)z$ and hence $x - kz \gtrsim z$. Suppose $z \succ 0$, whence $x - kz \succ 0$ and (since $]0, \infty[T \subset T$) $\lambda x - \mu z \in T$ for all $\lambda > 0 < \mu$. In particular, $\lambda x + (1-\lambda)(-z) \in T$ for all $\lambda \in]0,1[$, and consequently $-z \in \mathrm{lin}\, T$. Thus $-z \in (-T) \cap \mathrm{lin}\, T$, and since this is impossible by (iv) it follows that not $z \succ 0$. Hence (iv) implies (i) and the proof is complete.

An *n-dimensional utility function* for the preference relation $\gtrsim$ is a linear transformation v of E onto $\mathfrak{R}^n$ which satisfies the following two conditions for all $x, y \in E$:

(6) if $x \gtrsim y$, then $v(x) \gtrsim v(y)$;
(7) if $x \succ y$, then $v(x) \succ v(y)$.

Here the lexicographic ordering is employed in $\mathfrak{R}^n$[3].

When E is finite-dimensional (and conditions (1) and (2) are assumed), condition (3) guarantees the existence of a numerical utility function (Aumann [1]). This can be traced to the fact that finite-dimensionality of E is equivalent to idempotency of the 'lin' operation for convex sets [6]. When E is finite-dimensional, $\mathrm{lin}^2 X = \mathrm{lin}\, X$ for each convex $X \subset E$, and lin X is merely the closure of X in the natural topology of E. In the infinite-dimensional case, condition (3) loses much of its significance and must be replaced by explicit closure conditions in order to assure the existence of utility functions [5, 12].

Now suppose that dim $E = \aleph_0$, and let $\succsim$ be a relation on E satisfying condition (3). An example of Kannai [5] and Perles shows that the weak archimedean principle (3) (or, equivalently, the requirement that $(-T) \cap \text{lin}^1 T = \emptyset$) is not sufficient for the existence of a utility function. On the other hand, Kannai's main result asserts that the stronger archimedean principle, $(-T) \cap \text{lin}^{\Omega} T = \emptyset$, is sufficient.(1) From a construction in Section 2, it follows that Ω cannot be replaced by any countable ordinal. Indeed, for $1 < \beta < \Omega$ there exists an $\aleph_0$-dimensional preference relation $\succsim_\beta$ such that $(-T) \cap \text{lin}^{\alpha} T = \emptyset$ for all $\alpha < \beta$, and yet $\text{lin}^{\beta} T = E$. The latter condition implies that for each element y of the space E of alternatives and for each linear transformation v of E onto $\mathfrak{R}^n$, *every* element of $\mathfrak{R}^n$ appears as the value $v(x)$ for some alternative x which is preferred to y.(2)(3) Thus in a very strong sense, we may say that *no* useful information about $\succsim$ can be conveyed by means of v. (In particular, the only function v satisfying condition (6) is the identically zero function.)

2. **Iteration of the 'lin' operation.** When X is a convex subset of a linear space E, let us define the *order* of X (ord X) as the smallest ordinal number α such that $\text{lin}^{\alpha} X = \text{lin } X$ for all $\beta > \alpha$, and the *level* of X (lev X) as the set of all ordinal numbers β such that there exists a convex set C with $\text{lin}^{\beta} C = X$ but $\text{lin}^{\alpha} C \neq X$ if $\alpha < \beta$. It is known that ord $X \leqq \Omega$ [11, 13], and that dim $E \leqq \aleph_0$ if and only if ord $X < \Omega$ for all convex $X \subset E$; indeed, if dim $E = \aleph_0$ then every countable ordinal is realized as ord X for some convex $X \subset E$ [11, 14]. A new and simpler proof of the latter result is given below, and a question raised in [11] is answered by showing that lev E consists of all countable ordinals when E is $\aleph_0$-dimensional. In addition, the construction promised in Section 1 is carried out.

(1) Here Ω is the first uncountable cardinal. Kannai's Theorem B asserts that a utility function exists if $(-T) \cap cl\, T = \emptyset$, where $cl\, T$ is the closure of T in a certain topology τ_k for E. It is known [4, 7] that $\text{lin}^{\Omega} T$ is the closure of T in the *finite* topology τ_f for E, where a set is τ_f-open provided its intersection with each finite-dimensional flat F in E is open in the natural topology for F. But for dim $E \leqq \aleph_0$, it can be verified that Kannai's topology τ_k is identical with the finest locally convex topology τ_c for E, and it is known [4, 7] that τ_c is identical with τ_f.

(2) For let $T_y = \{x : x \succ y\}$. Then $T_y = T + y$, and hence vT_y is a subset of $\mathfrak{R}^n$ with $\text{lin}^{\beta} vT_y = \mathfrak{R}^n$. Since vT_y is convex, this implies that $vT_y = \mathfrak{R}^n$.

(3) For another example of this phenomenon, let E be the space of all (equivalence classes in the usual way, of) measurable functions on $[0, 1]$, topologized by means of the metric

$$\rho(x,y) = \int_0^1 \frac{|x(t) - y(t)|}{1 + |x(t) - y(t)|} dt$$

(corresponding to convergence in measure). Then dim $E = 2^{\aleph_0}$, but E is a complete separable metric linear space. Say that $x \succsim y$ provided $x(t) \geqq y(t)$ for almost all $t \in [0, 1]$. Then $\succsim$ is a pure preference relation, and in fact $(-T) \cap \text{lin}^{\Omega} T = \emptyset$. Nevertheless, $vT_y = \mathfrak{R}^n$ for each $y \in E$ and each linear transformation v of E onto $\mathfrak{R}^n$. This follows from the fact that E does not admit any nonzero linear form which is nonnegative on T [8].

A convex cone K will be called *proper* provided $K \cap -K \subset \{0\}$.

LEMMA. *If* $\dim E < \aleph_0$ *and the convex cone* K *in* E *is an* F_σ *set with* $0 \in K$, *then* K *is the union of an increasing sequence of closed convex cones.*

Proof. Let $L = K \cap -K$, a linear subspace of E, and let $K^+ = K \sim L$, a proper convex cone. Then K^+ is an F_σ set and hence is the union of an increasing sequence $Z_1 \subset Z_2 \subset \cdots$ of compact sets. For each i, the convex hull of Z_i is a compact convex subset of K^+ and hence the set $[0, \infty[$ con Z_i is a proper closed convex cone in K. For each i, let $J_i = L + [0, \infty[$ con Z_i. Clearly K is the union of the J_i's, and it can be verified that each J_i is closed. (Use 7.5 of [6] or 2.1 of [9]).

LEMMA *Suppose that* $\dim E = \aleph_0$, *and that* K *is an infinite-dimensional convex cone in* E *with* $0 \in K$. *Suppose that* K *is closed or that* K *is proper and an* F_σ *set (in the finite topology for* E*). Then there exist a linearly independent sequence* $b_1, b_2, \cdots$ *of points of* K *and an increasing sequence* $K_1 \subset K_2 \subset \cdots$ *of closed convex cones in* K *such that* $K = \bigcup_{i=1}^\infty K_i$ *and always*

$$\{b_1, \cdots, b_n\} \subset K_n \subset L_n,$$

where L_n is the linear hull of $\{b_1, \cdots, b_n\}$.

Proof. Let L denote the linear hull of K, whence K contains a basis $\{b_1, b_2, \cdots\}$ for L. For each n, let $C_n = K \cap L_n$. If K is closed, we simply take $K_n = C_n$. Suppose, then, that K is proper and is an F_σ set. It follows from the preceding lemma that for each n, C_n is the union of an increasing sequence $C_n^1 \subset C_n^2 \subset \cdots$ of closed convex cones such that $\{b_1, \cdots, b_n\} \subset C_n^1$. For each n, let

$$K_n = C_1^n + C_2^n + \cdots + C_n^n.$$

Then it is evident that K is the union of the K_i's, and since K is proper each of the cones K_n must be closed [9].

THEOREM. *Suppose that* E *and* K *are as in the preceding lemma. Then there exists a proper convex cone* K' *such that* K' *is an* F_σ *set,* $0 \in K' \subsetneqq K$, *and lin* $K' = K$.

Proof. Let the closed convex cones K_i be as in the lemma, and for each i let $K_i' = K_i +]0, \infty[b_{i+1}$. Let $K' = \{0\} \cup \bigcup_{i=1}^\infty K_i'$. Since $K_1' \subset K_2' \subset \cdots$ and since each set K_i' is a convex cone, it is evident that K' is a convex cone. Also, we have

$$K = \bigcup_{i=1}^{\infty} K_i \subset \bigcup_{i=1}^{\infty} \operatorname{lin} K_i' \subset \operatorname{lin} K',$$

and it remains only to show that lin $K' \subset K$. Consider an arbitrary point x of lin K'. There exists $y \in K'$ such that $]x, y] \subset K'$, and then for each i there exist $r(i)$, $k_i \in K_{r(i)}$, and $\tau_i > 0$ such that

$$\left(1 - \frac{1}{i}x\right) + \frac{1}{i}y = k_i + \tau_i b_{r(i)+1}.$$

Further, there exists n such that $\{x, y\} \subset L_n$, whence $r(i) < n$ for all i and $k_i + \tau_i b_{r(i)+1} \subset K_n$. Since K_n is closed, it follows that $x \in K_n \subset K$ and the proof is complete.

COROLLARY. *If X is an $\aleph_0$-dimensional convex F_σ set, then* lev X *includes all finite ordinal numbers.*

Proof. We may assume that the affine hull H of X is a hyperplane in $E \sim \{0\}$, where E is an $\aleph_0$-dimensional linear space. Let $K^0 = \{0\} \cup]0, \infty[X$, an F_σ proper cone in E, and consider an arbitrary finite β. By successive applications of the Theorem we can produce F_σ proper convex cones K^α such that always lin $K^\alpha = K^{\alpha-1}$ and

$$K^0 \underset{\neq}{\supset} K^1 \underset{\neq}{\supset} K^2 \underset{\neq}{\supset} \cdots \underset{\neq}{\subset} K^\beta.$$

Let $X^\beta = K^\beta \cap H$. Since $\text{lin}^\beta K^\beta = K^0$ but $\text{lin}^{\beta-1} K^\beta \neq K^0$, it follows that $\text{lin}^\beta X^\beta = X$ but $\text{lin}^{\beta-1} X^\beta \neq X$. Thus $\beta \in$ lev X and the proof is complete.

COROLLARY. *Suppose that X is an $\aleph_0$-dimensional convex F_σ set with $0 \in X$. If X is the direct sum of $\aleph_0$ isomorphs of X, then lev X consits of all countable ordinal numbers.*

Proof. Let $\mathscr{B}$ denote the set of all ordinal numbers β for which there exists a convex F_σ set Y having $\text{lin}^\beta Y = X$ but $\text{lin}^\alpha Y \neq X$ if $\alpha < \beta$. From the proof of the preceding corollary, it follows that $[1, \omega[\subset \mathscr{B}$ and that $\beta \in \mathscr{B}$ implies $\beta + 1 \in \mathscr{B}$. From a result in [11] (p. 233) in conjunction with the "direct sum" property of X, it follows that if $\beta < \Omega$ and $\alpha \in \mathscr{B}$ for all $\alpha < \beta$, then $\beta \in \mathscr{B}$. Hence lev $X = [0, \Omega[$ by transfinite induction.

COROLLARY. *If E is a linear space, then*

$$\text{lev } E = \begin{Bmatrix} \emptyset & \textit{if} \dim E < \aleph_0 \\ [1, \Omega[& \textit{if} \dim E = \aleph_0 \\ [1, \Omega] & \textit{if} \dim E > \aleph_0 \end{Bmatrix}.$$

COROLLARY. *Suppose that E is an $\aleph_0$-dimensional linear space and β is an ordinal number with $1 < \beta < \Omega$. Then there exists a pure preference relation $\succsim$ on E such that the convex cone $T = \{x : x \succ 0\}$ is an F_σ set and $\text{lin}^\beta T = E$, although $(-T) \cap \text{lin}^\alpha T = \emptyset$ for all $\alpha < \beta$.*

Proof. Let $\mathscr{B}$ denote the set of all ordinals $\beta \in]1, \Omega[$ for which the statement is true. We note first that $2 \in \mathscr{B}$. Indeed, let $\{b_1, b_2 \cdots\}$ be a basis for E and let K be the set of all points x of the form $x = \sum_{i=1}^n \lambda_i b_i$ with $\lambda_n > 0$. (That is, the last nonzero coordinate of x is positive.) Then K is a proper convex cone and is an F_σ set, so the Theorem guarantees the existence of an F_σ convex cone K' such that $\text{lin} K' = K \cup \{0\}$.

Let $x \gtrsim y$ provided $x - y \in K' \cup \{0\}$. Then $\gtrsim$ is a pure preference relation for which $T = K'$, $\mathrm{lin}^1 T = K \cup \{0\}$, and $\mathrm{lin}^2 T \supset \mathrm{lin}\ K = E$. It follows that $2 \in \mathscr{B}$.

Now suppose that $2 < \gamma < \Omega$ and that $\beta \in \mathscr{B}$ whenever $1 < \beta < \gamma$. If $\gamma - 1$ exists, then $\gamma - 1 \in \mathscr{B}$ and it follows from the Theorem that $\gamma \in \mathscr{B}$. Suppose, then, that γ is a limit ordinal, and for $1 < \beta < \gamma$ let E_β be an $\aleph_0$-dimensional linear space and $\gtrsim_\beta$ a pure preference relation on E_β such that T_β is an F_σ set, $\mathrm{lin}^\beta T_\beta = E_\beta$, and $(-T_\beta) \cap \mathrm{lin}^\alpha T_\beta = \emptyset$ for all $\alpha < \beta$. We may assume without loss of generality that E is the direct sum of the spaces E_β. Let T_γ be the direct sum of the convex cones T_β, and say that $x \gtrsim y$ provided $x - y \in T_\gamma \cup \{0\}$. Then the easily verified properties of T_γ show that $\gamma \in \mathscr{B}$.

It now follows by transfinite induction that $\mathscr{B} =]1, \Omega[$, so the proof is complete.

3. Unsolved problems. (a) If C is a convex set in an $\aleph_0$-dimensional linear space E, then lin C is an F_σ set. It follows, for a convex set $X \subset E$, that lev $X = \emptyset$ unless X is an F_σ set. We have seen that lev $X \supset [1, \omega[$ for every infinite-dimensional convex F_σ set in E, while for certain sets of this sort, lev $X = [1, \Omega[$. Is the latter equality valid for every $\aleph_0$-dimensional convex F_σ set X?

(b) Let E and T be as in footnote(3), so that $vT = F$ whenever v is a linear transformation of E onto a finite-dimensional linear space F. Is the same conclusion valid (for this particular choice of E and T) when $\dim F = \aleph_0$?(4)

References

1. R. J. Aumann, *Utility theory without the completeness axiom*, Econometrica **30** (1962), 445–462 and **32** (1964), 210–212.

2. P. C. Hammer, *Maximal convex sets*, Duke Math. J. **22** (1955), 103–106.

3. M. Hausner, Multidimensional utilities, in *Decision Processes*, edit. by Thrall, Coomb and Davis, Wiley, New York, (1954) 167–180.

4. S. Kakutani and V. Klee, *The finite topology of a linear space*, Arch. Math. **14** (1963) 55–58.

5. Y. Kannai, *Existence of a utility infinite dimensional partially ordered spaces*, Israel J. Math. **1** (1963), 229–234.

6. V. Klee, *Convex sets in linear spaces*, Duke Math. J. **18** (1951), 443–466.

7. ———, *Convex sets in linear spaces*. III, Duke Math. J. **20** (1953), 105–112.

(4) In this connection, the following consequence of a theorem in [10] (p. 58) may be useful If T is a convex cone in a linear space E and if E admits a linear transformation v onto a space F of countable dimension such that $vT \neq F$, then there is a linearly ordered set $(\Phi, \prec)$ of nonzero linear forms on E such that the following conditions are all satisfied:

(i) $1 \leqq \mathrm{card}\, \Phi \leqq \aleph_0$;
(ii) for each $x \in E$, $\varphi(x) = 0$ for all but finitely many $\varphi \in \Phi$;
(iii) for each $x \in T$, either $\varphi(x) = 0$ for all φ or there exists $\varphi' \in \Phi$ such that $\varphi'(x) > 0$ but $\varphi(x) = 0$ for all $\varphi \prec \varphi'$.

The theorem in [10] is concerned with the structure of *semispaces* [2, 10], a notion which may be employed to define various infinite-dimensional analogues of the lexicographic ordering in $\mathfrak{R}^n$.

8. ———, *Boundedness and continuity of linear functionals*, Duke Math. J. **22** (1955), 263–270.

9. ———, *Separation properties of convex cones*, Proc. Amer. Math. Soc. **6** (1955), 313-318.

10. ———, *The structure of semispaces*, Math. Scand. **4** (1956), 54–64.

11. ———, *Iteration of the "lin" operation for convex sets*, Math. Scand. **4** (1956), 231–238.

12. ———, *On representing a reference relation by means of a set u ility functions.* (to appear).

13. O. M. Nikodym, *On transfinite iterations of the weak linear closure of convex sets in linear spaces. Part A. Two notions of linear closure*, Rend. Circ. Mat. Palermo (ser. 2) **2** (1953,) 85–105.

14. ———, *On transfinite iterations of the weak linear closure of convex sets in linear spaces Part B. An existence theorem in weak linear closure*, Rend. Circ. Mat. Palermo (ser. 2) **3** (1954), 5–75.

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